Unit-VII :

Laplace T/F's - Introduction.

Definition of Laplace transforms.

s-plane (or) complex-plane.

Convergence conditions for Laplace T/F's.

ROC : Region of Convergence and

Properties of ROC

Relationship b/w Fourier T/F & Laplace T/F.

Uni-lateral LT/F and it's Convergence

Conditions

Inverse Laplace T/F (methods to findout Inverse LT/F's)

Properties of LT/F's. — Unilateral & Bilateral

Introduction to L T/F:-

The Fourier T/F is a tool which allows us to represent a signal x(t) as a continuous sum of exponentials in the form $e^{j\omega t}$, whose frequencies are restricted to the imaginary-axis in the complex-plane $s = j\omega$. Such a representation is quite valuable in the analysis, however, the use of Fourier T/F leaves much to be desired.

1. The F T/F exists only for a restricted class of signals and therefore, cannot be used for such i/p's as growing exponentials.

2. The F T/F can not be used easily to analyze unstable (or) even marginally stable systems.

The Laplace T/F:-

The basic reason for both these difficulties is that for some signals, such as $e^{at}u(t)$ $(a>0)$. The Fourier T/F does not exist because ordinary sinusoids (or) exponentials of the form $e^{j\omega t}$ (on account of their constant amplitudes) are incapable of synthesizing exponentially growing signals. This Problem could be resolved if it were Possible to use basis signals of the form e^{st} (instead of $e^{j\omega t}$), where the Complex frequency S is not restricted to just the imaginary-axis (as in the F T/F). This is Precisely what is done in the following extended T/F Known as the "Bilateral Laplace T/F".

Where The freq-variable $s=j\omega$ is generalized to $s=\sigma+j\omega$. Such generalization Permits Us to Use exponentially growing sinusoids to synthesize a signal $x(t)$.

Before, developing the mathematical Operations required for Such an extension. We will find it illuminating to have an intutive Understanding Of such a generalization

Intuitive Understanding Of the Laplace T/F:

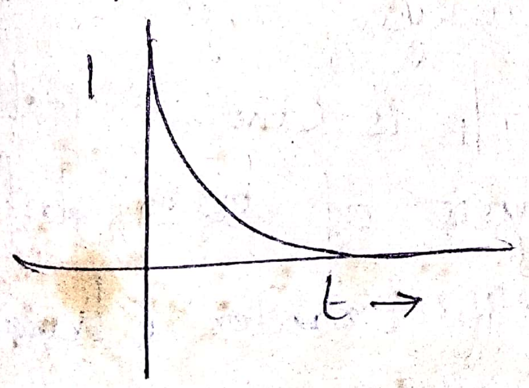
If a signal $x(t)$ is not Fourier T/F mable we may be able to make it Fourier T/Fmable by multiplying it with a decaying exponential such as $e^{-\sigma t}$.

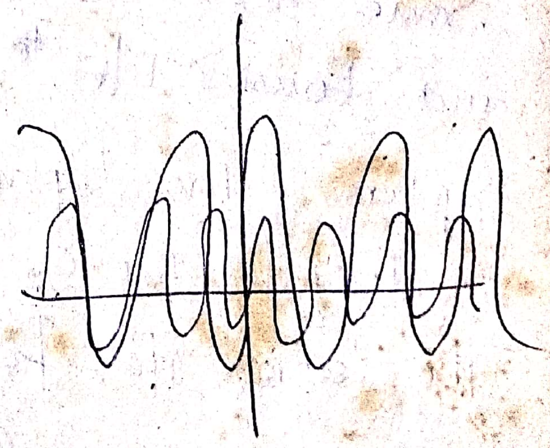
for eg: a signal $e^{2t}u(t)$ Can be made Fourier T/F mable by multiplying it with $e^{-\sigma t}$ with $\sigma > 2$. let $\phi(t) = f(t)e^{-\sigma t}$. Now the Signal $\phi(t)$ is now Fourier T/F mable and it's Fourier Components are of the form $e^{j\omega t}$ with frequencies ω varying from $\omega = -\infty$ to ∞

The exp. Components $e^{j\omega t}$ and $e^{-j\omega t}$ in the spectrum add to give a sinusoidal frequency ω. The spectrum Contains an infinite no. of Such sinusoids each having infinitesimal amplitude

It would be very confusing to draw all these components

$$\phi(t) = f(t)e^{-\sigma t}$$





Laplace T/F's :-

The L T/F is widely used to solve Linear differential eqn's and Corresponding Initial and final-value Problems.

L T/F's are used to analyse CT S&S.

It is a mathematical tool that transforms any CT signal into a completely different signal representation i.e a function of a complex variable s. One major advantage of L T/F is that the initial conditions are taken care of input in the begining. They are also used in Electrical Ckt Analysis.

=> In previous chapters we have discussed Some mathematical tools such as Fourier Series and Fourier T/F to analysis Signals and Systems. The Laplace T/F is another tool which is used for the analysis of S&S. Infact, The Laplace T/F provides broader characterization of the Signals and Systems compared to Fourier T/F.

i.e, In some cases, Laplace T/F can be used where FT/F can not be used.

Laplace T/F Can be used for the analysis of Unstable Systems Where as F T/F has several limitations.

An important difference b/w The FT/F and the $\mathcal{L}$ T/F is that the F T/F uses a summation of waves of +ve and -ve frequencies where as the LT/F employs damped waves through the Use of an additional factor $e^{-\sigma}$ where as σ is the +ve number.

Both FT/F and LT/F are mathematical Operations which convert The time domain function $x(t)$ to the freq-domain func^n $X(e^{j\omega})$ and $X(s)$. Also, the LT/F provides The total solution to The diff. eq^n and The Corresponding Initial and final value Problems.

The Laplace T/F is an important and Powerful mathematical tool in the system analysis and design. LT/F is widely used for describing the Continuous Circuits and Systems including automatic Control systems and also analyse

Signal flow Through the Causal Linear-Time Invariant Systems With Non-Zero initial - conditions. Also the Z- T/F to be Suitable for dealing With discrete signal && systems.

Few points about L T/F :-

* Fourier T/F represents CT Signal in-terms of complex Sinusoidals i.e $e^{j\omega t}$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

The L T/F Provides broader Characterization compared to F T/F.

* L T/F represents CT Signals in-terms of complex exponentials. i.e e^{-st} Hence L T/F is used to analyze the signals (or) functions which are not absolutely integrable.

* CT Systems are also analyzed more effectively using L T/F's Can be applied to the analysis of unstable systems also.

* L T/F of the impulse Response is Called System function (or) T.F. Many Properties

of the system such as Causality, Stability invertibility freq-Response etc. Can be studied with the help of L T/F.

The L T/F Comes in 2 varieties.

(i). Unilateral (or) One-sided.

(ii). Bilateral (or) Two-sided.

Unilateral is a convenient tool for solving differential eqn's with initial conditions. The Bilateral T/F offers insight into the nature of the system characteristics such as stability, Causality and freq-response.

The Primary role of LT/F in engineering is transient and stability analysis of Causal LTI system described by Diff. eqns.

Advantages of L T/F:-

There are several advantages for LT/F's Compared to Other informations.

1. The Performance of LTI System Can be measured in-terms of Diff. eqns.

2. It can be used to analyze the signals which are not absolutely integrable.

3. The transformation is used to represent function by sum of exp. of the form e^{st}.

4. The Bi-lateral L T/F is used to measure The Performance of the system in-terms of causality & stability.

Limitations of L T/F :-

* Freq- Response of the System Can not be drawn (or) estimated. indeed only the Pole - zero plot Can be drawn.

* $S = j\omega$ is used Only for sinusoidal steady- state analysis.

Applications of Laplace T/F's :-

By using L T/F method, The Transient Currents in electrical Ckts. Containing energy storage elements Can be determined -easily.

Definition of Laplace T/F :-

The F T/F of a CT signal is represented as $F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ —①—

Where $x(t)$ is $= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$ —②.

Now, let $s = \sigma + j\omega$ a Complex Variable and by replacing $j\omega$ by $\sigma + j\omega$.

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt.$$

Let $s = \sigma + j\omega$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt} \quad \text{—(I)—}$$

and the above eqn is known as L T/F of a Signal $x(t)$ and is Bi-lateral LT/F. it exists for $-\infty$ to ∞.

Consider the F T/F of the Signal $x(t)\, e^{-\sigma t}$

$$F\{x(t)\, e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= X(\sigma + j\omega)$$

$$F\{x(t)\, e^{-\sigma t}\} \longleftrightarrow X(\sigma + j\omega).$$

So, the inverse F T/F of the signal $X(\sigma+j\omega)$ is

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)\, e^{j\omega t}\, d\omega.$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)\, e^{j\omega t} \cdot e^{\sigma t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega)\, e^{(\sigma+j\omega)t}\, d\omega.$$

We know that $S=\sigma+j\omega$.

as $\omega \to -\infty$ $\quad s \to \sigma - j\infty$

$\omega \to +\infty$ $\quad s \to \sigma + j\infty$

and $ds = j\,d\omega$.

for a given $x(t)$, σ has a certain minimum value σ_0 and we can select any value of $\sigma > \sigma_0$.

let σ is defined as c then. $c > \sigma_0$.

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds.} \quad \text{— II}$$

The pair of eqn's I & II is known as Bi-lateral L T/F pair (or) 2 sided L T/F pair

i.e as $X(s) = L\{x(t)\}$ $\quad x(t) = L^{-1}\{X(s)\}$

Scanned with OKEN Scanner

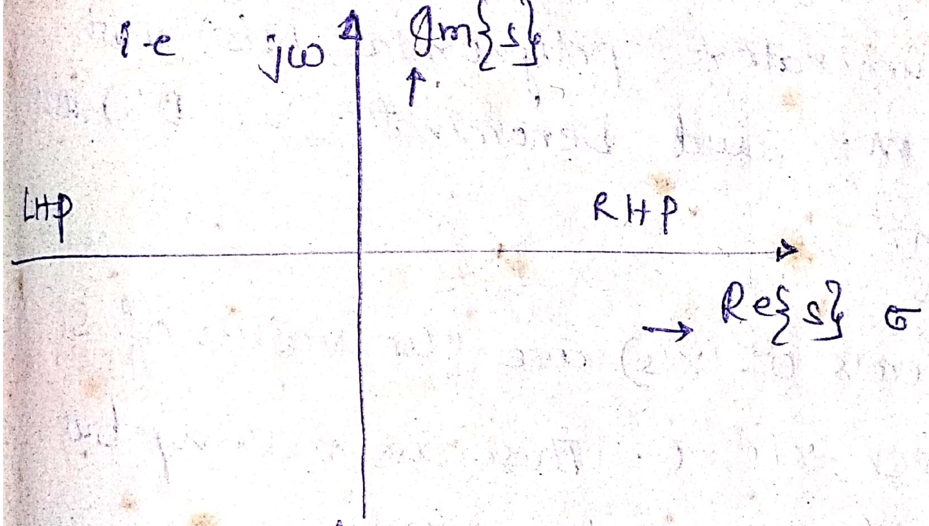
(or) Simply as $x(t) \Leftrightarrow X(s)$.

S-plane:-

The Laplace variable s is a complex-variable defined by $s = \sigma + j\omega$. Where σ is the Real Part of s denoted as $Re\{s\}$ and $j\omega$ is the imaginary Part of s as $Im\{s\}$

So the range of variation of s can be represented schematically as a plane known as S-plane.

i.e



s-plane.

The Horizontal axis is σ-axis and Vertical axis is $j\omega$-axis

The Origin is represented as a Complex zero ie $0 + j0$.

The region to the left of the imaginary-axis $(-\infty$ to $0)$ is known as

Left Half plane. and the region to The Right of the $j\omega$ axis as (0 to ∞) Right Half plane (LHP and RHP).

* S-plane is a complex plane.
* The Most general form of LT/F is a ratio of 2 polynomials (i.e. Rational function)

as $X(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$

The Numerator polynomial N(s) with degree m. and Denominator " D(s) with degree n.

The Zero's of X(s) are the values of s for which X(s)=0. These are nothing but the roots of the Polynomial N(s)=0

i.e as

$$(s-z_1)(s-z_2) \dots \dots (s-z_m) = 0$$

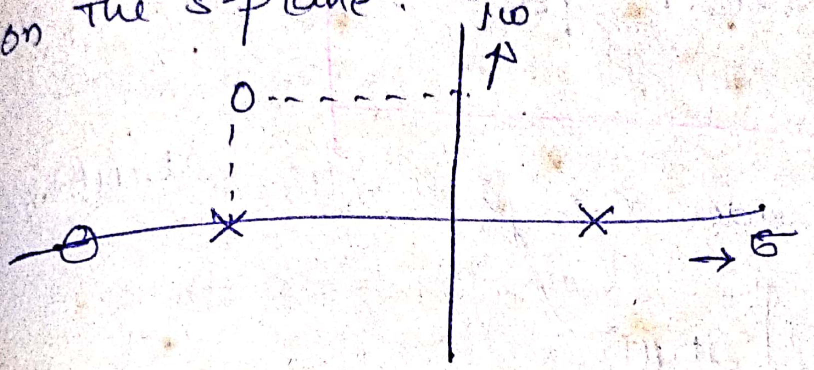
$$\text{i.e } N(s) = b_m \sum_{k=1}^{m} (s-z_k) = 0$$

The poles are the values of s for which X(s) becomes infinite. and are the roots of the Polynomial D(s)=0

$$\prod_{k=1}^{n} (s-p_k) = 0$$

$$(s-p_1)(s-p_2)\cdots(s-p_n)=0$$

We denote zero's by O and Poles by X on the s-plane.



(Pole-zero plot).

Convergence of L T/F:-

The Bilateral L T/F of a signal $x(t)$ exists if $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ is finite.

where $s = \sigma + j\omega$

i.e $X(\sigma+j\omega) = \int_{-\infty}^{\infty} [x(t)\, e^{-\sigma t}] e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \phi(t)\, e^{-j\omega t}\, dt.$$

From the above '2' eqn's note that L T/F is nothing but the F T/F of the signal $\phi(t) = x(t) e^{-\sigma t}$.

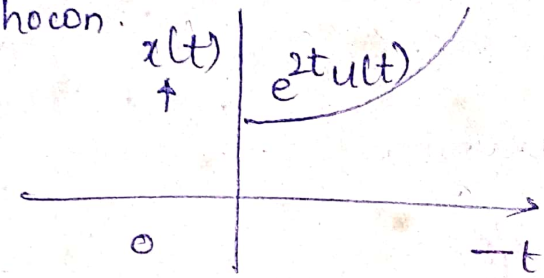
So a necessary condition for the

convergence of $\mathcal{L}$ T/F is it's absolute integrability of the function $\phi(t)$

i.e $X(s)$ exists only if $\int_{-\infty}^{\infty} |\phi(t)| < \infty$

i.e, $\boxed{\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, dt < \infty}$

i.e consider a signal $x(t) = e^{2t} u(t)$ as shown.



For This signal the Fourier T/F does not exists.

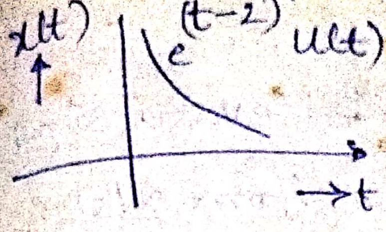
if is not absolutely integrable.

Now. By multiplying the signal. $x(t)$ with Then The resultant signal is $x(t)\, e^{-\sigma t}$.

$\therefore e^{(2-\sigma)t} u(t)$ is the resultant signal.

which is absolutely integrable for $\sigma > 2$.

$\therefore$ The Laplace T/F is Nothing but the Fourier T/F of the signal $x(t)\, e^{-\sigma t}$. exists for some value of σ.



$\sigma > 2$

Thus, we can conclude that the Laplace T/F exists for signals for which F.T/F does not exists.

If $x(t)$ is absolutely integrable, then the F T/F of $x(t)$ can be obtained from it's L-T/F as

$$\boxed{X(j\omega) = X(s)\Big|_{s=j\omega}}$$

Region of convergence (ROC):-

The Range of values of the complex variable s for which L T/F $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ converges is called the Region of convergence (ROC).

i.e The Region of convergence (or) Existence of signal's Laplace T/F $X(s)$ is the set of values of s for which the integral defining the direct L T/F $X(s)$ converges.

Role of ROC:-

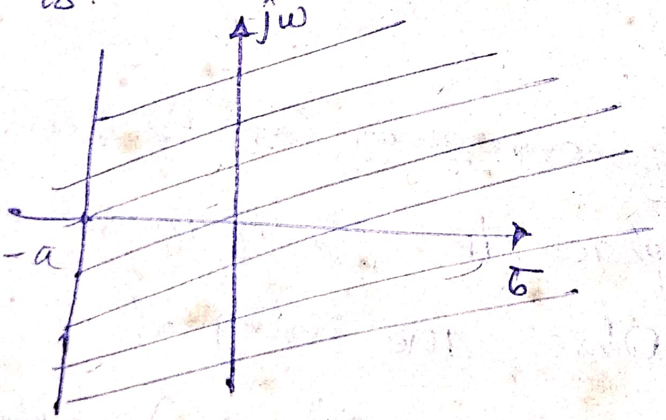
The ROC is required for evaluating the inverse L T/F of $x(t)$ from $X(s)$.

i.e The operation of finding the inverse T/F requires an integration in the complex plane

i.e $x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma} X(s) e^{st} ds$.

The path of integration is along s-plane $s = \sigma + j\omega$, i.e along $\sigma + j\omega$, with ω varying from $-\infty$ to ∞ and Moreover, The Path of integration must lie in the ROC for $X(s)$.

For the Signal $e^{-at}u(t)$, This is Possible if $\sigma > -a$ so the path of integration is.



Thus to obtain $x(t) = e^{-at}u(t)$ from $X(s) = \frac{1}{s+a}$, the integration is

Performed through this path. for the funcn $\frac{1}{(s+a)}$. Such integration in the complex plane requires a background in the Theory of functions of complex variables.

So we can avoid this integration by compiling a Table of $\mathcal{L}$ T/F's. So for Inverse $\mathcal{L}$ T/F's we use this ~~table~~ instead of performing complex integration.

Specific constraints on the ROC are closely associated with time-domain properties of $x(t)$.

Properties of ROC / constraints (or) Limitations

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

i.e The ROC of $X(s)$ consists of the values of s for which F T/F of $x(t)e^{-\sigma t}$ converg. This is possible if $x(t)e^{-\sigma t}$ is fully integrable. Thus the condition depends only on σ.

Hence ROC is the strips (bands) which is only in terms of values of σ.

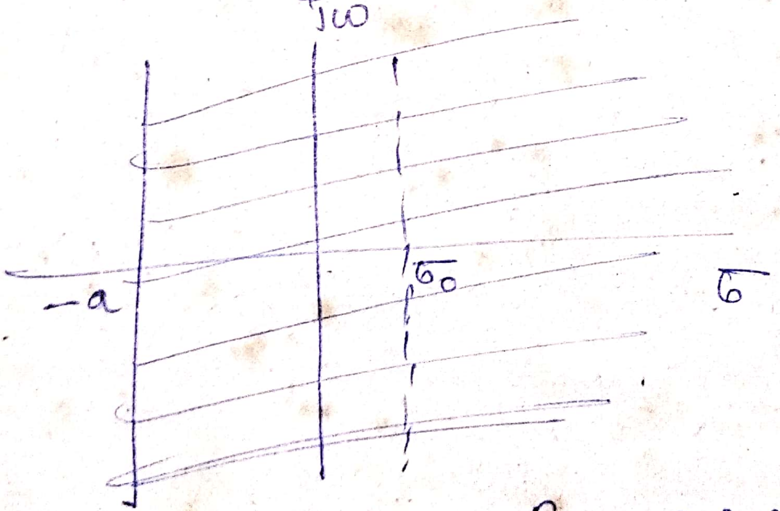
3. For Rational Laplace T/F's, The ROC does not contain any Poles.
[This is because $X(s)$ is infinite at poles and the integral can not be converge at this pt].

4. If $x(t)$ is of finite duration, and absolutely integrable; then the ROC is the entire s-plane.

5. If $x(t)$ is right-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, Then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.

i.e if the signal is $x(t) = e^{-at}u(t)$ right-sided [0 to ∞] Then $X(s) = \frac{1}{s+a}$ for

ROC: $Re\{s\} > -a$



ROC: $Re\{s\} > -a$.

6. If $x(t)$ is left-sided and if the line $Re\{s\}=\sigma_0$ The ROC, Then all values of s for which $Re\{s\}<\sigma_0$ will also be in the ROC.

7. If $x(t)$ is two-sided, and if the line $Re\{s\}=\sigma_0$ is in the ROC Then The ROC consists of a strip in the S-plane, that includes the line $Re\{s\}=\sigma_0$.

For the both sided sequence, The ROC lies in the Region $\sigma_1 < Re\{s\} < \sigma_2$. This ROC is the strip parallel to $j\omega$ axis in The S-plane.

(8). If the L T/F $X(s)$ of $x(t)$ is rational, Then it's ROC is bounded by Poles (or) extends to infinity in addition no Poles of $X(s)$ are contained in the ROC.

If the function has '2' poles, Then ROC will be area b/w these '2' Poles for 2 sided sequence. if for single sided sequence The area extends from One pole to infinity.

But it does not include any pole.

(9). If the L T/F $X(s)$ of $x(t)$ is rational, Then if $x(t)$ is right-sided. The ROC is the region in the S-plane to the right of the right most pole and if $x(t)$ is left-sided, the ROC is the region in the s-plane to the left of the left-most Pole.

Relation b/w Laplace and Fourier T/F's :-

We know that Fourier T/F is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{—①}$$

Fourier T/F exists only if $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

We know that $s = \sigma + j\omega$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, e^{-j\omega t}\, dt \quad \text{—②}$$

if we compare these '2' eqn's

both are equal if $\sigma = 0$, $\boxed{X(s) = X(j\omega)\Big|_{s=j\omega}}$

This means Laplace T/F is same as FT/F when $s = j\omega$

∴ Fourier T/F is nothing but the special-case of L T/F where $s = j\omega$ indicates the imaginary axis in complex-plane.

Thus LT/F is basically Fourier T/F on imaginary axis in the S-plane.

Connection to The Fourier T/F :-

When $s = j\omega$ L T/F is F T/F.

i.e $X(j\omega) = X(s)$. When s is replaced by $s = j\omega$

we know that Fourier T/F of $x(t) = e^{-at}u(t)$

as $\frac{1}{a+j\omega}$. Replacing $j\omega$ with s.

it is $\frac{1}{s+a} = X(s)$.

Unfortunately it is not valid for all $x(t)$.

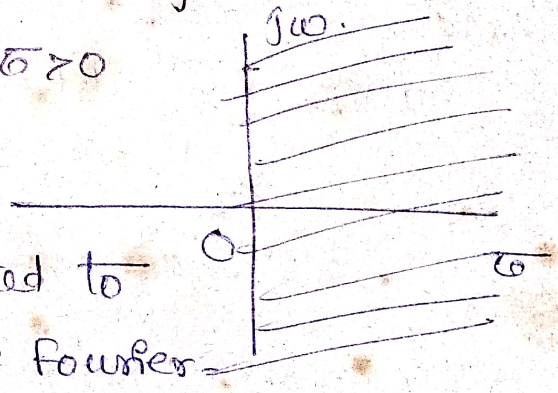
We may use it Only if the ROC for $X(s)$ includes $j\omega$ axis.

i.e for example The $\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega}$

But $\mathcal{L}\{u(t)\} = \frac{1}{s}$ and ROC: $Re\{s\} > 0$.

ROC, Which does not include $j\omega$ axis.

i.e $Re\{s\} > 0 \Rightarrow \sigma > 0$



ROC is $\sigma > 0$.

The reason for this complication is related to the Convergence of the Fourier - Integral.

Where the path of the integration is restricted to imaginary axis.

Because of this reason. The Fourier Integral for the step function does not converge in the Ordinary sense.

So, we had to use a generalized (impulse) function for convergence.

→ The Laplace integral converges for u(t) only for $Re\{s\} > 0$.

Another interesting fact is that although the $\mathcal{L}$ T/F is a generalization of the Fourier T/F. There are signals (periodic signals) for which $\mathcal{L}$ T/F does not exist, although the Fourier T/F exists (but not in the ordinary sense).

Uni-lateral Laplace T/F

In order to understand the need for defining a Uni-lateral Transform. Let us find.

$$x(t) = -e^{-at}u(-t) \qquad , \qquad x(t) = e^{-at}u(t)$$

$$= \frac{1}{s+a} \text{ for } Re\{s\} < -a. \qquad = \frac{1}{s+a}, \text{ and } Re\{s\} > -a$$

The given '2' signals has same Laplace T/F's except the ROC's.

i.e While finding inverse $\mathcal{L}$T/F's

There is a possibility of having more than one inverse T/F. ~~depen~~ Unless until ROC is not specified.

i.e There is no One-to-one Correspondence b/w $X(s) \longleftrightarrow x(t)$. Unless ROC is not specified.

This fact increases the complexity in using L T/F.

This complexity arises to handle Causal Non-Causal signals with Unique nature.

i.e., if we restrict our signals to the Causal type, such an ambiguity does not arise.

i.e if $\frac{1}{s+a}$ is given $\longleftrightarrow e^{-at}u(t)$.

i.e the Uni-lateral (One-sided) T/F is a special case of the Bi-lateral T/F.

if we restrict all signals as Causal type. the limits of integration becomes 0 to ∞.

Existence of Unilateral T/F:-

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

0^- → indicates the initial condition just before the appⁿ of i/p

0^+ → " " just after the appⁿ of i/p.

→ Unilateral → is used to solve linear differential eqn's with initial conditions

⇒ Unilateral LT/F Cannot analyze Non-Causal Systems for Non-Causal i/p's.

⇒ If signal itself is causal (i.e $x(t)=0$, t
Unilateral T/F = Bilateral T/F.
and. ROC: RH of s-plane.

⇒ diffⁿ property is different for both Uni-lateral & Bi-lateral T/F.

⇒ In general → LT/F means Uni-lateral LT/F.

Inverse Laplace T/F's :-

If $X(s)$ is given finding $x(t)$ from $X(s)$ is known as finding it's Inverse Laplace T/F's.

The methods to find it out are

1. Use of Tables →
2. Partial fraction expansion →
3. Using Inversion formula (or) Contour Integration
4. Convolution Integral.

Inversion formula:

⇒ This will holds for all classes of T/F

⇒ and x(t) is obtained by evaluating line-integral in the complex-s-plane.

i.e $x(t) = \frac{1}{2\pi j} \oint_C X(s) e^{st} ds$.

By compiling Tables for finding Laplace T/F's of various signals and tabulate the values as $x(t) \xleftrightarrow{LT} X(s)$

By dividing the given X(s) into '2' functions

as $X(s) = F_1(s) \cdot F_2(s)$.

and finding $F_1(s) \to f_1(t)$

$F_2(s) \to f_2(t)$

$X(s) = f_1(t) * f_2(t)$

Now perform convolution for the '2' signals $f_1(t)$ and $f_2(t)$

and the resultant signal is it's (Time-domain)

Inverse Fourier T/F.

Properties of Laplace Transforms (Bilateral)

1. **Linearity :-**

if $x_1(t) \longleftrightarrow X_1(s)$ ROC: R_1

$x_2(t) \longleftrightarrow X_2(s)$ ROC: R_2

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(s) + a_2 X_2(s)$$

ROC: $R_1 \cap R_2$.

2. **Time-shifting :-** $x(t) \longleftrightarrow X(s)$ ROC: R

$x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$. ROC: R.

$$L\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st} dt$$

$t - t_0 = \lambda$

$dt = d\lambda$

$$= \int_{-\infty}^{\infty} x(\lambda)\, e^{-st_0} \cdot e^{-s\lambda} d\lambda$$

$$= e^{-st_0} \left[\int_{-\infty}^{\infty} x(\lambda)\, e^{-s\lambda} d\lambda \right]$$

$$= e^{-st_0} X(s).$$

$x(t) = e^{-2t} u(t) \longleftrightarrow \dfrac{1}{s+2}$ ROC: $Re\{s\} > -2$

$x(t-1) = e^{-2(t-1)} u(t-1)$

$$L\{x(t-1)\} = \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1)\, e^{-st} dt.$$

$$= \int_{1}^{\infty} e^{-2t} e^{2} e^{-st} dt$$

$$= e^{2}\int_{1}^{\infty} e^{-(2+s)t}\,dt \qquad ROC: R$$

$$\sigma > -2$$

$$= e^{2}\left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_{1}^{\infty}$$

$$= e^{2}\left[\frac{e^{-(2+s)}}{+(2+s)}\right]$$

$$= \frac{\cancel{e^{2}}\cdot \cancel{e^{-2}}\, e^{-s}}{(2+s)}$$

$$= \frac{e^{-s}}{s+2}$$

3. Shifting in s-domain

$$x(t) \longleftrightarrow X(s) \qquad ROC: R$$

$$e^{s_0 t}x(t) \longleftrightarrow X(s-s_0) \qquad ROC: R+Re\{s_0\}$$

if $x(t) = e^{-2t}u(t)$

$$e^{5t}x(t) \longleftrightarrow ?$$

$$\mathcal{L}\{e^{5t}x(t)\} = \{e^{5t}\cdot e^{-2t}u(t)\}$$

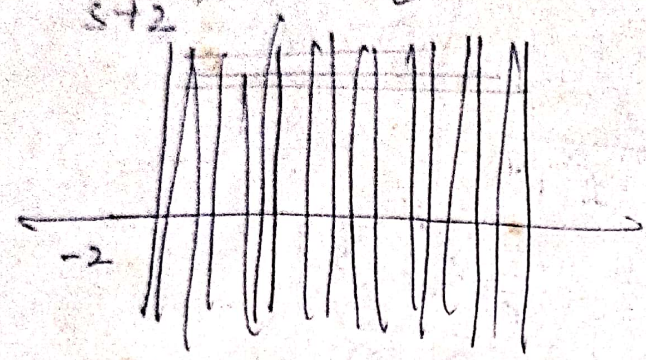
$$= e^{+3t}u(t)$$

$$= \frac{1}{s-3} \qquad Re\{s\} > 3$$

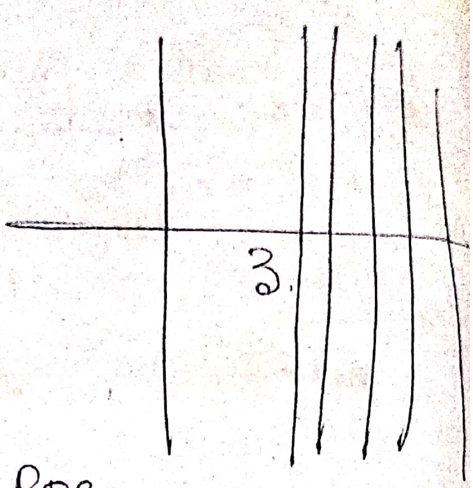
$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$$

$$e^{5t}\cdot e^{-2t}u(t) \longleftrightarrow \frac{1}{(s-5)+2} = \frac{1}{s-3}$$

$\frac{1}{s+2}$, ROC: $\sigma > -2$



$\frac{1}{s-3}$. ROC: $\sigma > 3$.



ROC: $\sigma > (-2+5)$

> 3

$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

$e^{(5+j3)t}\, e^{-2t} u(t) \longleftrightarrow \frac{1}{[s-(5+j3)]+2}$

$= \frac{1}{(s-5+2)-j3}$

$= \frac{1}{s-3-3j}$

$= \frac{1}{\sigma+j\omega-3-3j}$

$= \frac{1}{(\sigma-3)+j(\omega-3)}$

$= \frac{1}{s-3}$.

Integration in time domain (Bi-lateral)

$x(t) \longleftrightarrow X(s)$

$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s}$

$\Rightarrow \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$

$= \int_{-\infty}^{t} [x(t) * u(t)]$

$L\{\int_{-\infty}^{t} x(\tau)\,d\tau\} = \frac{X(s)}{s}$

for multiple integrals

$\int_{-\infty}^{t}\int_{-\infty}^{t} \dots x(t)\,dt_1 dt_2 \dots dt_n = \frac{X(s)}{s^n}$

Integration in s-domain

$x(t) \longleftrightarrow X(s)$

$\frac{-x(t)}{t} \longleftrightarrow \int_{s}^{\infty} X(s)\,ds$

$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}dt$

$\int_{s}^{\infty} X(s)\,ds = \underbrace{\int_{-\infty}^{\infty} \frac{-x(t)}{t} e^{-st} ds}_{L \cdot T/F}$

Differentiation in Time-domain (Bi-lateral)

$x(t) \xleftrightarrow{LT/F} X(s)$ / ROC: R

$\frac{d\,x(t)}{dt} \longleftrightarrow sX(s)$ / ROC: R

we know that $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}ds$

$\frac{d\,x(t)}{dt} = \underbrace{\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\,X(s)\,e^{st}ds}_{L \cdot T/F}$

$$\boxed{\begin{array}{l} \frac{d\,x(t)}{dt} \longleftrightarrow sX(s) \\ \frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s);\ \text{ROC: R} \end{array}}$$

Differentiation in s-domain:

$x(t) \longleftrightarrow X(s)$, ROC: R

$-x(t)\,t \longleftrightarrow \frac{dX(s)}{ds}$, ROC: R

$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}dt$

$\frac{dX(s)}{ds} = \underbrace{\int_{-\infty}^{\infty} -t\,x(t)\,e^{-st}dt}$

$$\boxed{\begin{array}{l} \therefore \frac{dX(s)}{ds} \longleftrightarrow -t\,x(t) \\ \frac{d^n X(s)}{ds^n} \longleftrightarrow (-t)^n x(t) \\ \text{ROC: R} \end{array}}$$

Time-Reversal :-

$x(t) \xleftrightarrow{LT/F} X(s)$ ROC: R

$x(-t) \longleftrightarrow ?$ ROC: $-R$.

let $-t = \lambda$

$-dt = d\lambda$

$t: \to -\infty$ to ∞

$\lambda: \to \infty$ to $-\infty$

$$L\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)\, e^{-st}\, dt$$

$$= \int_{\infty}^{-\infty} x(\lambda)\, e^{s\lambda} (-d\lambda)$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, e^{s\lambda}\, d\lambda$$

$$= X(-s)$$

$$\boxed{x(-t) \longleftrightarrow X(-s)}$$

Eg: $x(t) = e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$ ROC : $\sigma > -1$;

$x(-t) = e^{t}u(-t) \longleftrightarrow \frac{-1}{s-1}$ ROC: $\sigma < 1$

$-\sigma > -1$

$\boxed{\sigma < 1.}$

Time-scaling

$x(t) = e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$ $\sigma > -2$

$x(2t) = e^{-4t}u(2t) \longleftrightarrow \frac{1}{s+4}$ $\sigma > -4$

$$\int_{-\infty}^{\infty} x(2t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(4+s)t}\, dt$$

$$= \frac{1}{s+4}$$

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$

ROC: $\frac{R}{a}$

$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$ ROC: $\sigma > a$

$x(-at) \longleftrightarrow \frac{1}{a} X\left(\frac{-s}{a}\right)$ $\neq -1$

Differentiation Property for Uni-lateral L.T.

$$x(t) \xleftrightarrow{U.L.T/F} X(s)$$

$$\frac{dx(t)}{dt} \longleftrightarrow$$

$$L\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$= \left[e^{-st} \int \frac{dx(t)}{dt}\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} (-s) e^{-st} \int \frac{dx(t)}{dt} dt$$

$$= \left[e^{-st} x(t)\right]_{0}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$= -x(0^-) + sX(s)$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)$$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - s^{n-1} x(t)\Big|_{t=0^-} - s^{n-2} \frac{dx(t)}{dt}\Big|_{t=0^-} - \cdots$$

$$\cdots\; s^{n-n} \frac{d^{n-1} x(t)}{dt^{n-1}}\Big|_{t=0^-}$$

Integration in Time domain — (Uni-lateral T/F)

$$x(t) \longleftrightarrow X(s)$$

$$\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow$$

$$L\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = \int_{0^-}^{\infty} \int_{-\infty}^{t} x(\tau) d\tau \, e^{-st} dt$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau) d\tau = \int_{-\infty}^{0^-} x(\tau) d\tau + \int_{0^-}^{t} x(\tau) d\tau$$

$$L\left\{\int_{0^-}^{t} x(\tau) d\tau\right\} = \int_{0^-}^{\infty} \int_{0^-}^{t} x(\tau) d\tau \, e^{-st} dt$$

$$= \left[\int_0^t x(\tau)d\tau \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} d\left[\int_0^t x(\tau)d\tau\right]$$

$$= \left[0 - \int_0^0 x(\tau)d\tau e^{-0}\right] + \frac{1}{s}\int_0^\infty e^{-st}x(t)dt$$

$$= \frac{X(s)}{s}$$

$$L\left\{\int_{-\infty}^{0} x(\tau)d\tau\right\} = \int_0^\infty \int_{-\infty}^{0^-} x(\tau)d\tau\, e^{-st}dt$$

$$= \int_{-\infty}^{0^-} x(\tau)d\tau \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau$$

$$\boxed{L\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau}$$

Convolution in time-domain:-

if $x_1(t) \longleftrightarrow X_1(s)$ ROC: R_1

$x_2(t) \longleftrightarrow X_2(s)$ ROC: R_2

$x_1(t) * x_2(t) \longleftrightarrow X_1(s).X_2(s)$, ROC: $R_1 \cap R_2$

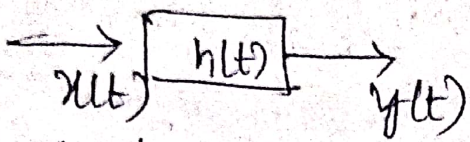
$$\mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau).\left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt\right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-s\tau} d\tau \; X_2(s)$$

$$= X_1(s) \cdot X_2(s).$$

Importance of Convolution property:-



To find impulse response h(t), and also to find o/p $y(t) = h(t) * x(t)$

$$= L^{-1}\{ H(s) X(s) \}.$$

Causality:- For a Causal LTI system, The impulse response is zero for $t<0$ [i.e $h(t)=0,\ t<0$]

The Roc associated with the system function for a Causal system is a right-half plane.

But the Converse of this is not necessarily true. i.e, an Roc to the right of the right most pole doesnot guarantee that a system is Causal. rather it guarantees only the impulse response is right-sided.

For a system with a rational system function Causality of the system is equivalent to the Roc being the right-half plane to the right of the right most pole.

(i). $h(t) = e^{-t}u(t)$ — $h(t)=0$, for $t<0$. This is Causal.

$H(s) = \frac{1}{s+1}$ $Re\{s\} > -1$

is rational. Roc is right of the right most Pole (-1) so it is Causal.

(ii). if $h(t) = e^{-|t|}$. — $h(t) \neq 0$, $t<0$. → System is non-Causal.

$H(s) = \frac{-2}{s^2-1}$.

Roc: $-1 < Re\{s\} < 1$

$H(s)$ is rational and Roc is not the right of the right most pole (1). but it is bounded b/w -1 and 1. ∴ The System is Non-Causal.

(iii). $H(s) = \frac{e^s}{s+1}$, $Re\{s\} > -1$

Here Roc is right of the right most pole. and $H(s)$ is irrational.

$h(t) = e^{-(t+1)}u(t+1)$

$h(t) \neq 0$. for $t<0$

∴ The System is Non-Causal.

i.e Causality implies that the Roc is to the right of the right most pole but the Converse may not be true. in general. unless the system func is rational.

Stability :- The ROC of H(s) can also be related to the stability of a system.

The stability of an LTI system is equivalent to its impulse response h(t) being absolutely integrable.

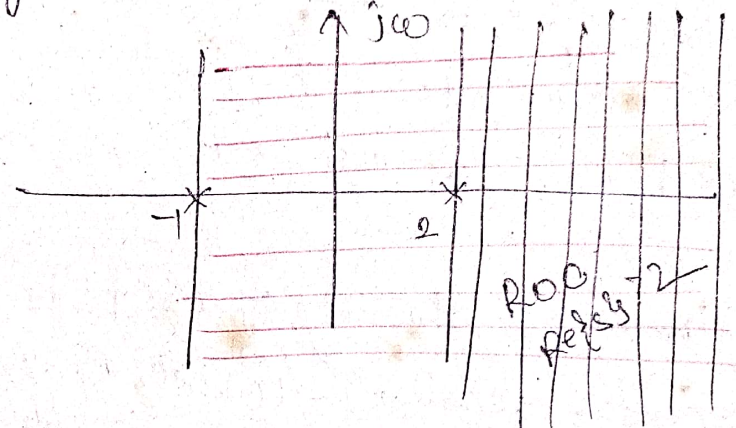
An LTI system is stable if and only if the ROC of its system funcn. H(s) includes $j\omega$-axis. i.e $Re\{s\} = 0$.

eg:- i.e $H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{2}{3(s+1)} + \frac{1}{3(s-2)}$

$h(t) = \frac{2}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$. $\left(\int_{-\infty}^{\infty} |h(t)|dt\right) < \infty$

$Re\{s\} > -1$ $Re\{s\} > 2$.

overall ROC is $Re\{s\} > 2$ which doesn't include $j\omega$ axis. So system is unstable.



for the system to be stable

$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$.

$Re\{s\} > -1$ $Re\{s\} < 2$.

$\Rightarrow$ $-1 < Re\{s\} < 2$

and h(t) is absolutely integrable.

For a system to be stable (or unstable) and have a system function that is not rational. For example, $H(s) = \frac{e^s}{s+1}$. The system is not rational. and it's impulse response.

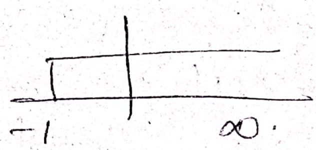
$h(t) = e^{-(t+1)} u(t+1)$ is absolutely integrable. indicating that the system is stable.

i.e $\int_{-\infty}^{\infty} |h(t)| \, dt$

$u(t+1) = 1, \quad t+1 \geq 0$

$t \geq -1$

$\Rightarrow \int_{-1}^{\infty} e^{-(t+1)} dt$



$\Rightarrow e^{-(t+1)} < \infty$.

However, for systems with rational functions stability is easily interpreted in-terms of the poles of the system.

A Causal system with rational T.F $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left-half of s-plane. i.e all of the poles have -ve real parts.

1) Check the stability of a CT LTI system with Transfer function which is given by.

$$H(s) = \frac{s-2}{(s+2)(s-3)}$$

$$= \frac{A}{s+2} + \frac{B}{s-3}$$

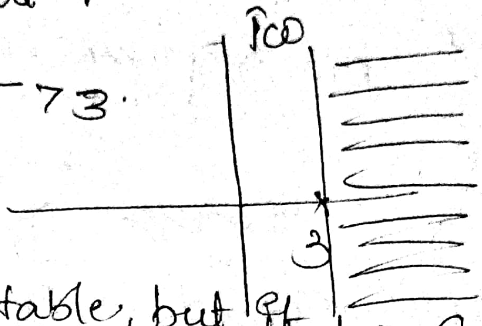
$$= \frac{4}{5(s+2)} + \frac{1}{5(s-3)}$$

$h(t) = \frac{4}{5} e^{-2t} u(t) + \frac{1}{5} e^{3t} u(t)$ → if we choose the 2 signals as right sided.

$Re\{s\} > -2$ $\sigma > 3$.

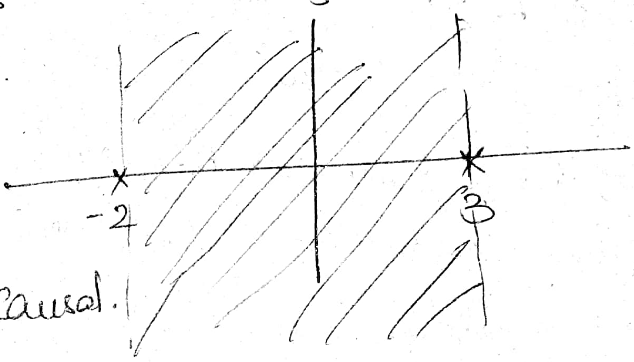
∴ RoC: $Re\{s\} > 3$.

∴ system becomes unstable, but it be Causal.



Case 2:- if $h(t) = \frac{4}{5} e^{-2t} u(t) - \frac{1}{5} e^{3t} u(-t)$

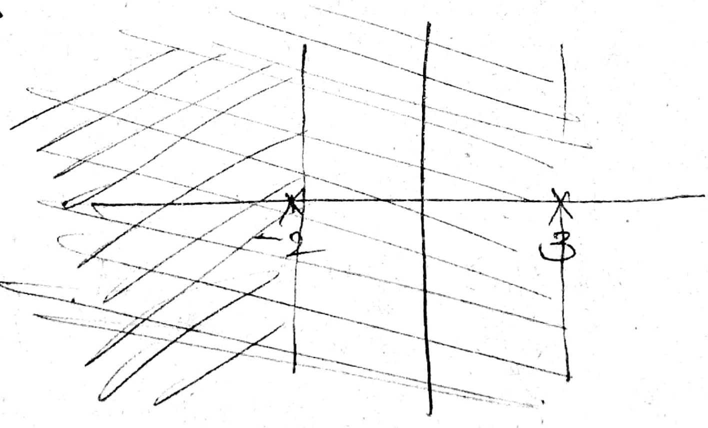
If RoC is bounded and includes jω axis system is stable. but Non-Causal.



RoC: $-2 < Re\{s\} < 3$.

Case 3:- $h(t) = -\frac{4}{5} e^{-2t} u(-t) - \frac{1}{5} e^{3t} u(-t)$

RoC ~~not~~ does not include jω axis and is anti-causal and unstable



RoC: $Re\{s\} < -2$

Solution of Differential equations using LT/F

We know that the total response of a system can be expressed as the sum of two components.

i.e Total response = Natural Response + Forced response.

Natural response of the system is due to initial conditions of the system. whereas the forced response is due to i/p alone.

The L.T/F of the system is used for solving differential equations offers clear seperation b/w the Natural Response of the system to initial conditions and the forced response of the system associated with the input.

i.e.
$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_0 y(t)$$
$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_0 x(t). \quad -①-$$

Using time-Diffn property

$$\frac{d^N y(t)}{dt^N} = s^N Y(s) \quad \text{and} \quad \frac{d^M x(t)}{dt^M} = s^M X(s)$$

This shows that x(t) and y(t) has no initial conditions.

i.e
$$y(0^-) = \frac{dy(0^-)}{dt} = \cdots = 0$$
$$x(0^-) = \frac{dx(0^-)}{dt} = \cdots = 0.$$

Scanned with OKEN Scanner

$\therefore$ eqn (1) becomes.

$$a_N s^N Y(s) + a_{N-1} s^{N-1} Y(s) + \cdots + a_0 Y(s)$$
$$= [b_M s^M + b_{M-1} s^{M-1} + \cdots + b_0) X(s)$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{b_N s^M + b_{N-1} s^{M-1} + \cdots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0}$$

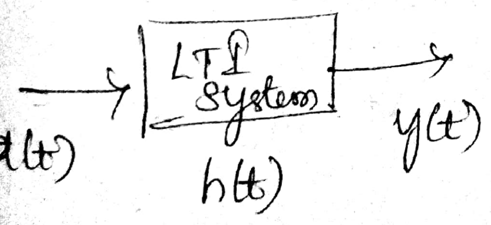
$Y(s)$ is zero state response (or) Forced response

[without initial conditions].

$H(s)$ depends upon the Coefficients of Differential equation. It does not depend on the i/p signal (or) the initial energy storage.

Analysis and Characterization of LTI systems using the Laplace T/F

One of the important application of the Laplace transform is in the analyses and characterization of LTI systems.



$$Y(s) = H(s) X(s)$$

Specifically, The Laplace T/F of the i/p and o/p of an LTI system are related through multiplication by the Laplace T/F of the impulse response of the System.

$X(s)$ - LT/F of i/p.

$Y(s)$ - L " o/p.

$H(s)$ - The impulse response of the system.

On the response of LTI systems to complex exponentials if the input to an LTI system is $x(t) = e^{st}$. Then o/p will be $H(s)e^{st}$.

i.e e^{st} is an eigen function of the system with eigen values equal to the LT/F of the impulse response.

i.e. $y(t) = x(t) * h(t)$ $\qquad x(t) = e^{st}$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$\boxed{y(t) = e^{st} \cdot H(s)}$$

For $s = j\omega$, $H(s)$ is the Frequency-response of the LTI System. $H(s)$ is Commonly Known as

System Function (or) T/F. Many properties of the LTI systems can be closely associated with the characteristics of the system funcn. in s-plane.

* Differentiation property for nth order is

$$L\left\{\frac{d f^{n}(t)}{dt^{n}}\right\} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0).$$

Find the Laplace T/F of $\left[\frac{d^2}{dt^2}\delta(t)\right]$

$$L\left\{\frac{d^2}{dt^2}\delta(t)\right\} = s^2 L\{\delta(t)\} - s^1\delta(0) - s^{2-2}\delta'(0).$$

$$= s^2(1) - s(1) - 0$$

$$= s^2 - s$$

(1) Find the Uni-lateral LT/F of $x(t) = \{e^{-3t}u(t) * tu(t)\}$

$$L\{x(t)\} = L\{e^{-3t}u(t) * tu(t)\}$$ $\{x_1(t) * x_2(t)\} \Rightarrow X_1(s)X_2(s)$

$$= \frac{1}{s+3} \times \frac{1}{s^2}$$

$$= \frac{1}{s^2(s+3)}.$$

(2) Find Uni-lateral L T/F of $x(t) = e^{-t}(t-2)u(t-2)$.

We know that $tu(t) \longleftrightarrow 1/s \quad Re\{s\} > 0$

$x(t) \longleftrightarrow X(s)$

$x(t-2) \longleftrightarrow e^{-2s}X(s)$

$\therefore (t-2)u(t-2) \longleftrightarrow \frac{e^{-2s}}{s^2}$

Then by using frequency-translation property

$e^{-t}\{(t-2)u(t-2)\} \longleftrightarrow e^{-t}\left\{\frac{e^{-2s}}{s^2}\right\}$

$\longleftrightarrow \frac{e^{-2(s+1)}}{(s+1)^2}$

→ Verify the Diff^n property for the signal $x(t) = e^{at}u(t)$. of Uni-lateral T/F.

$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)$

$x(t) = e^{at}u(t) = e^{at}$ for $t \geq 0$. $\rightarrow x(0^+) = 1$

$\frac{dx(t)}{dt} = a \cdot e^{at}$.

$L\{e^{at}u(t)\} \longleftrightarrow \frac{1}{s-a}$ for $Re\{s\} > a$.

$L\left\{\frac{dx(t)}{dt}\right\} \longleftrightarrow sX(s) - x(0^+)$

$\longleftrightarrow \frac{s}{s-a} - 1$

$\longleftrightarrow \frac{\cancel{s} - \cancel{s} + a}{s-a}$

$\longleftrightarrow \frac{a}{s-a}$. — (2)

$\frac{dx(t)}{dt} = ae^{at} \longleftrightarrow \frac{a}{s-a}$. — (1)

(1) = (2).

$\therefore \frac{dx(t)}{dt} = sX(s) - x(0^+)$

→ Use integration property to show that the unilateral LT of $t\,u(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$ is given by $1/s^2$

Integration property is

$$\int_{-\infty}^{t} u(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s} + \frac{x^{-1}(0^+)}{s}$$

where $x^{-1}(0^+) = \int_{-\infty}^{0^+} x(\tau)\,d\tau$ is the area under $x(t)$ from $t=-\infty$ to $t=0^+$.

$$\Rightarrow \int_{-\infty}^{t} u(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^+} u(\tau)\,d\tau}{s}$$

$$\longleftrightarrow \frac{1}{s^2} + \frac{\int_{-\infty}^{0^+} u(\tau)\,d\tau}{s}$$

$$\int_{-\infty}^{0^+} u(\tau)\,d\tau = 0 \quad [\because u(\tau) \text{ exists for +ve portion}].$$

$$\int_{-\infty}^{t} u(\tau)\,d\tau \longleftrightarrow \frac{1}{s^2}$$

$$\cancel{\Rightarrow \int_{-\infty}^{t} u(\tau)\,d\tau \Rightarrow [u(\tau)\int d\tau - 0]_{-\infty}^{t}}$$

$$\cancel{\Rightarrow u(t)\,t}$$

Similarly $L\{t\,u(t)\} \Rightarrow$

$$\Rightarrow \int_{0}^{\infty} t\,e^{-st}\,dt \Rightarrow \left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$\Rightarrow 0 - 0 + 1/s^2 \Rightarrow 1/s^2$$

$$\therefore L\{t\,u(t)\} = L\left\{\int_{-\infty}^{t} u(\tau)\,d\tau\right\} = 1/s^2.$$

→ Use LT/F of find the o/p of the system described by the differential equation $\frac{dy(t)}{dt} + 5y(t) = x(t)$ in response to the i/p $x(t) = 3e^{-2t}u(t)$ and initial condition $y(0^+) = -2$.

$$L\left\{\frac{dy(t)}{dt} + 5y(t)\right\} = L\{x(t)\}.$$

$$s.Y(s) - y(0^+) + 5Y(s) = L\{3e^{-2t}u(t)\}$$

$$(s+5)Y(s) + 2 = \frac{3}{s+2}$$

$$(s+5)Y(s) = \frac{3}{s+2} - 2$$

$$= \frac{3 - 2s - 4}{(s+2)}$$

$$Y(s) = \frac{-(2s+1)}{(s+2)(s+5)}$$

$$= \frac{-1}{(s+2)} - \frac{3}{(s+5)}$$

$$A = \frac{-(-4+1)}{(-2+5)}$$

$$= -3/3 = -1$$

$$B = \frac{-(-10+1)}{(-5+2)}$$

$$= -\frac{(-9)}{+3} = -3$$

$$y(t) = -e^{-2t}u(t) - 3e^{-5t}u(t).$$

Differentiation in t-domain for uni-lateral:

if $x(t) \longleftrightarrow X(s)$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)$$

i.e $$L\left\{\frac{dx(t)}{dt}\right\} = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt.$$

$= \; e^{-st} \;\}$

since unilateral T/F

$$= \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

$$\int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} \rightarrow e^{-st} \int_{0^+}^{\infty} \frac{dx(t)}{dt} dt + \int_{0^+}^{\infty} \left[s e^{-st} \int_{0^+} \frac{dx(t)}{dt} dt \right] dt$$

$$\Rightarrow \left[e^{-st} x(t) \right]_{0^-}^{\infty} + s \cdot \int_{0^+}^{\infty} e^{-st} x(t) dt$$

$$\Rightarrow e^{-\infty} x(\infty) - x(0^-) + sX(s)$$

$$\Rightarrow sX(s) - x(0^-)$$

$$\boxed{\therefore \; L\left\{ \frac{dx(t)}{dt} \right\} = sX(s) - x(0^-)}$$

i.e for n^{th} order differential, it is

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s) - s^{n-1} x(t)\Big|_{t=0^+} - s^{n-2} \frac{dx(t)}{dt}\Big|_{t=0^+} \cdots$$

$$- s^{n-3} \frac{d^2 x(t)}{dt^2}\Big|_{t=0^+} \cdots - s^2 \frac{d^{n-3} x(t)}{dt^{n-3}}\Big|_{t=0^+} \cdots$$

$$- s^1 \frac{d^{n-2}}{dt^{n-2}} x(t)\Big|_{t=0^+} - s^0 \frac{d^{n-1}}{dt^{n-1}} x(t)\Big|_{t=0^+}.$$

$$\leftrightarrow s^n X(s) - s^{n-1} x(0^+) - s^{n-2} x'(0^+) - s^{n-3} x^2(0^+)$$

$$\cdots s^2 x^{n-3}(0^+) - s^1 x^{n-2}(0^+) \cdots - x^{n-1}(0^+)$$

→ find Laplace T/F of $t\cos\omega_0 t\, u(t)$, $t\sin\omega_0 t\, u(t)$

$$x(t) = t\sin\omega_0 t\, u(t)$$

$$= t\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]u(t)$$

$$= \frac{1}{2j}\left[t e^{j\omega_0 t}u(t) - t e^{-j\omega_0 t}u(t)\right]$$

$$X(s) = \frac{1}{2j}\left[L\{t e^{j\omega_0 t}u(t)\} - L\{t e^{-j\omega_0 t}u(t)\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(s-j\omega_0)^2} - \frac{1}{(s+j\omega_0)^2}\right]$$

$$= \frac{1}{2j}\left[\frac{(s+j\omega_0)^2 - (s-j\omega_0)^2}{[s^2+\omega_0^2]^2}\right]$$

$$= \frac{1}{\cancel{2j}} \frac{\cancel{4}^{2}\cancel{j}\omega_0 s}{(s^2+\omega_0^2)^2}$$

$$X(s) = \frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$$

$$x(t) = t\cos\omega_0 t\, u(t)$$

$$= t\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]u(t)$$

$$= \frac{1}{2}\left[t e^{j\omega_0 t}u(t) + t e^{-j\omega_0 t}u(t)\right]$$

$$X(s) = \frac{1}{2}\left[L\{t e^{j\omega_0 t}u(t)\} + L\{t e^{-j\omega_0 t}u(t)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-j\omega_0)^2} + \frac{1}{(s+j\omega_0)^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(s^2 - \omega_0^2)}{(s^2+\omega_0^2)^2}\right]$$

$$X(s) = \frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$$

→ find $L\{\sin(at+\theta)\,u(t)\}$

$$x(t) = \sin(at+\theta)\,u(t)$$

$$x(t) = \sin at\cos\theta\, u(t) + \cos at\sin\theta\, u(t)$$

$$\Rightarrow X(s) = \cos\theta\, L\{\sin at\, u(t)\} + \sin\theta\, L\{\cos at\, u(t)\}$$

$$= \cos\theta \frac{a}{s^2+a^2} + \sin\theta \cdot \frac{s}{s^2+a^2}$$

$\rightarrow x(t) = e^{-at}u(t)$ find $X(s)$

$$X(s) = \int_0^\infty e^{-at} e^{-st} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

if $s+a<0$. the $-(s+a)$ becomes +ve $\therefore e^{-(s+a)\infty}$ becomes ∞

$\therefore$ for L. T/F to exists :

$\Rightarrow (s+a) > 0$

$\Rightarrow s > -a \Rightarrow Re\{s\} > -a$

$$X(s) = \frac{1}{s+a}$$

$$\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} > -a}$$

$x(t) = e^{at}u(t)$

$$X(s) = \int_0^\infty e^{at} e^{-st} dt$$

$$= \int_0^\infty e^{(a-s)t} dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$a-s<0 \Rightarrow \frac{-1}{a-s} \Rightarrow \frac{1}{s-a}$

$s > a$

$$\boxed{e^{at}u(t) \leftrightarrow \frac{1}{s-a}, \quad Re\{s\} > a}$$

Scanned with OKEN Scanner

$\rightarrow x(t) = e^{at}u(-t)$

$$X(s) = \int_{-\infty}^{0} e^{(a-s)t}\,dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)}\right]_{-\infty}^{0}$$

$a-s>0$

$a>s \qquad = \frac{1}{a-s}$

$s<a$

$Re\{s\}<a \qquad = \frac{-1}{s-a}$

$$e^{at}u(-t) \longleftrightarrow -1/s-a, \; Re\{s\}<a$$

$\rightarrow x(t) = e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{0} e^{-(a+s)t}\,dt$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_{-\infty}^{0}$$

$$= \frac{-1}{s+a}, \quad a+s<0$$

$s<-a$

$Re\{s\}<-a$

$$e^{-at}u(-t) \longleftrightarrow \frac{-1}{s+a}, \; Re\{s\}<-a$$

** Basic exponentials

$e^{-at}u(t) \rightarrow \frac{1}{s+a} \longrightarrow Re\{s\}>-a$

$e^{at}u(t) \rightarrow \frac{1}{s-a} \longrightarrow Re\{s\}>a$

$e^{at}u(-t) \rightarrow \frac{-1}{s-a} \rightarrow Re\{s\}<a$

$e^{-at}u(-t) \rightarrow \frac{-1}{s+a} \longrightarrow Re\{s\}<-a$

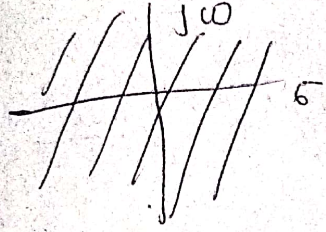
1) $L\{\delta(t)\}$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

$$= e^{-st}\Big|_{t=0}$$

$$= 1$$

ROC: entire s-plane.



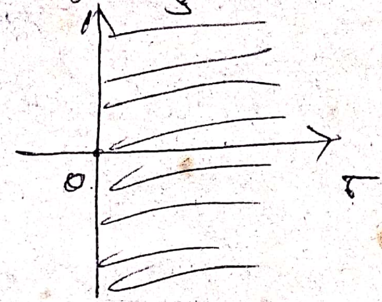
2) $u(t) = 1$, $t \geq 0$

$= 0$ otherwise.

$$L\{u(t)\} = \int_0^{\infty} e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{s}, \quad Re\{s\} > 0$$



3) $r(t) = t$, $t \geq 0$

$= 0$ otherwise

$$L\{r(t)\} = \int_0^{\infty} t e^{-st} dt$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$\boxed{t \Leftrightarrow 1/s^2, \quad Re\{s\} > 0}$$

4) $e^{j\omega_0 t} u(t)$

$$\int_0^{\infty} e^{-(s - j\omega_0) t} dt$$

$$= \frac{1}{s - j\omega_0}$$

$Re\{s\} > j\omega_0$

$\sigma > j\omega_0$

$$e^{-j\omega_0 t} u(t) \Leftrightarrow \frac{1}{s + j\omega_0}, \quad Re\{s\} > -j\omega_0$$

Initial Value Theorem -

The initial value theorem allows us to Calculate initial value of $x(t)$. i.e, $x(0)$ directly from the T/F $X(s)$. without the need for finding the inverse T/F of $X(s)$.

$$\text{i.e } L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

from the definition of $L\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt$
Uni-lateral.

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0^-)$$

Now apply $\underset{s\to\infty}{Lt}$ on both sides.

$$\underset{s\to\infty}{Lt} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \underset{s\to\infty}{Lt} \{sX(s) - x(0^-)\}$$

$$\Rightarrow 0 = \underset{s\to\infty}{Lt}\, sX(s) - x(0^-)$$

i.e if $x(t)$ is continuous at the origin. (i.e at $t=0$), $\frac{dx(t)}{dt}$ does not contain an impulse term at $t=0$. and as $s\to\infty$. The integral goes to zero.

$$\therefore \underset{s\to\infty}{Lt}\, sX(s) = x(0^-) = x(0^+)$$

$$\boxed{x(0^+) = \underset{t\to 0^+}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)}$$

Final Value Theorem:-

The value of $x(t)$ as t tends to infinity may also be found directly from the Laplace T/F $X(s)$

i.e $L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^+)$

$$\underset{s\to 0}{Lt} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \underset{s\to 0}{Lt} \left[sX(s) - x(0^-)\right]$$

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} dt = \underset{s\to 0}{Lt}\, sX(s) - x(0^-)$$

$$\left[x(t)\right]_{0^-}^{\infty} = \underset{s\to 0}{Lt}\, sX(s) - x(0^-)$$

$$\Rightarrow x(\infty) - \cancel{x(0^-)} = \underset{s\to 0}{Lt}\, sX(s) - \cancel{x(0^-)}$$

$$\Rightarrow \boxed{x(\infty) = \underset{s\to 0}{Lt}\, sX(s) = \underset{t\to\infty}{Lt}\, x(t)}$$

If $sX(s)$ has all its singularities in the left-half of s-plane, then $\underset{s\to 0}{Lt}\, sX(s)$ exists.

→ A simple pole in $X(s)$ at the origin is permitted otherwise all other poles of $X(s)$ must be strictly in the left-half of s-plane.

→ To apply Final value theorem, we must check whether (or) not there are common factors b/w the numerator and Denominator of $sX(s)$.

If yes, we cancel the common factors and then check for the poles of the $sX(s)$. If any of the poles lies in R.H.S p-lane, then the final-value theorem does not hold.

1) Determine the initial and final values of a signal $x(t)$ whose uni-lateral L.T/F is

$$X(s) = \frac{7s+10}{s(s+2)}$$

Initial-value $x(0^+) = \underset{s\to\infty}{Lt}\ sX(s)$

$$= \underset{s\to\infty}{Lt}\ \frac{(7s+10)\cancel{s}}{\cancel{s}(s+2)}$$

$$= \underset{1/s\to 0}{Lt}\ \frac{7+10/s}{1+2/s}$$

$$= 7$$

Final value is $x(\infty) = \underset{s\to 0}{Lt}\ sX(s)$

$$= \underset{s\to 0}{Lt}\ \frac{7s+10}{s+2}$$

$$= 10/2$$

$$= 5$$

We may verify the result using inverse L.T/F

$$X(s) = \frac{7s+10}{s(s+2)} = A/s + \frac{B}{s+2}$$

$$= \frac{5}{s} + \frac{2}{s+2}$$

$A = \frac{10}{2} = 5$

$B = \frac{-14+10}{-2} = \frac{-4}{-2} = 2$

Scanned with OKEN Scanner

$x(t) = 5u(t) + 2e^{-2t}u(t)$

$x(0^+) = 5u(0) + 2e^{-0}u(0)$

$= 5+2$

$x(0^+) = 7$

$x(\infty) = 5u(\infty) + 2e^{-\infty}u(\infty)$

$= 5+0$

$= 5$

→ The response of LTI system for $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt}$, $y(0^-)=2$, $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$, $x(t) = u(t)$ will be

$x(0^+) = u(0^+) = 1$

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt}$$

$$\frac{d^ny(t)}{dt^n} = s^nY(s) - s^{n-1}y(0) - s^{n-2}\frac{dy}{dt}(0^-) - s^{n-3}y''(0) \dots$$

$$\dots\dots y^n(0)$$

$\Rightarrow s^2Y(s) - sy(0^-) - \left.\frac{dy(t)}{dt}\right|_{t=0^-} + 2sY(s) - y(0^-)$

$+5Y(s) = sX(s) - x(0^-)$

$\Rightarrow s^2Y(s) - 2s - 0 + 2sY(s) - 2 + 5Y(s) = sX(s)$

$\Rightarrow (s^2+2s+5)Y(s) = sX(s) + 2s + 2$

$(s^2+2s+5)Y(s) = 2(s+1)$

$$Y(s) = \frac{2(s+1)+?}{s^2+2s+5} =$$

$$Y(s) = \frac{2(s+1)+1}{s^2+2s+5} = \frac{\cancel{2}(s+1)}{(\cancel{s+5})(s-3)}$$

$$Y(s) = \frac{2(s+1)+1}{s^2+2s+1+4} \Rightarrow \frac{2(s+1)}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}$$

$$y(t) = 2 \cdot e^{-t}\cos 2t\, u(t) + \frac{1}{2}e^{-t}\sin 2t\, u(t)$$

→ Find $x(t)$, if $X(s) = \frac{1}{(2s+1)^2+4}$

$$X(s) = \frac{1}{4[(s+1/2)^2+4]}$$

$$X(s) = \frac{1/4}{(s+1/2)^2+4}$$

$$x(t) = \frac{1}{4}e^{-t/2}\cos 2t\, u(t)$$

→ The corresponding time signal for the bilateral Laplace transform $X(s) = \left(\frac{1}{s} - \frac{e^{-s}}{s} - e^{-2s}\right)$ with ROC: $Re\{s\} < 0$ is

$$X(s) = \frac{1}{s} - \frac{e^{-s}}{s} - e^{-2s}, \qquad \delta(t) \quad Re\{s\} < 0$$

$$= -u(-t) + u(-t+1) - \delta(t+2)$$

$1 \longleftrightarrow \delta(t)$ for leftsided

$1 \longleftrightarrow \delta(-t)$

$e^{-2s} \longleftrightarrow \delta(-t-2)$

$\longleftrightarrow \delta[-(t+2)]$

$\delta(t) = \delta(-t)$ even $= \delta(t+2)$

$$\int_{-\infty}^{\infty} \delta(t+2)\, e^{-st}\, dt$$

$$= e^{-st}\Big|_{t=-2}$$

$$= e^{+2s}$$

→ The response of LTI system for $\frac{dy(t)}{dt}+10y(t)=10x(t)$, $y(0^-)=1$, $x(t)=u(t)$ will be.

$$\frac{dy(t)}{dt}+10y(t)=10x(t) \qquad y(0^-)=1, \quad x(t)=u(t)$$

$$sY(s)-y(0^-)+10Y(s)=10X(s)$$

$$(s+10)Y(s)-1=\frac{10}{s}$$

$$(s+10)\,Y(s)=\left(\frac{10}{s}+1\right)$$

$$Y(s)=\frac{\cancel{(s+10)}}{s\cancel{(s+10)}}$$

$$Y(s)=1/s$$

$$\boxed{y(t)=u(t)}$$

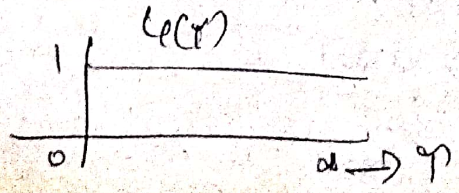
→ $y(t)=u(t)*h(t)$ where $h(t)=\begin{cases} e^{2t}, & t<0 \\ e^{-3t}, & t>0 \end{cases}$

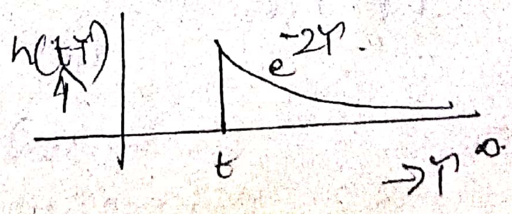
$$y(t)=\int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau=\int_{-\infty}^{\infty} h(\tau)\,u(t-\tau)\,d\tau$$

$$=\int_{-\infty}^{0} e^{2\tau}u(t-\tau)\,d\tau+\int_{0}^{\infty} e^{-3\tau}u(t-\tau)\,d\tau$$

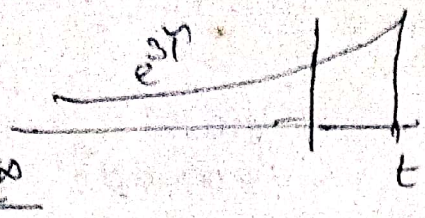
(or) $$\int_{-\infty}^{0} u(\tau)\,e^{2(t-\tau)}\,d\tau+\int_{0}^{\infty} e^{-3(t-\tau)}u(\tau)\,d\tau$$

$$h(t-\tau)=e^{2(t-\tau)}, \quad t-\tau<0 \Rightarrow t<\tau \Rightarrow \tau>t$$





$= e^{-3(t-\tau)}$ for $t-\tau>0 \Rightarrow t>\tau, \Rightarrow \tau<t$



$\tau \to -\infty$

$$\Rightarrow \int_{\tau=t}^{\infty} e^{+2t} e^{-2\tau}\, d\tau + \int_{\tau=0}^{t} e^{-3t} e^{3\tau}\, d\tau$$

$$\Rightarrow e^{2t}\left[\frac{e^{-2\tau}}{-2}\right]_t^{\infty} + e^{-3t}\left[\frac{e^{3\tau}}{3}\right]_0^{t}$$

$$\Rightarrow e^{2t}\left[\frac{e^{-2t}}{2}\right] + e^{-3t}\left[\frac{e^{3t}-1}{3}\right]$$

$$\boxed{y(t) \Rightarrow \frac{1}{2} + \frac{1}{3}\left[1-e^{-3t}\right]}$$

in an alternate form.

add $\frac{1}{2}e^{2t} - \frac{1}{2}e^{2t}$ to the above equation

$$\Rightarrow \frac{1}{2}e^{2t} - \frac{1}{2}e^{2t} + \frac{1}{2} + \frac{1}{3}\left[1-e^{-3t}\right]$$

$$\boxed{y(t) \Rightarrow \frac{1}{2}e^{2t} + \frac{1}{6}\left[5 - 2e^{-3t} - 3e^{2t}\right].}$$

$\rightarrow$ The Time signal $x(t)$ corresponding to

$$X(s) = s\frac{d^2}{ds^2}\left(\frac{1}{s^2+9}\right) + 1/_{s+3}$$

We know that

$$x(t) \longleftrightarrow X(s)$$

$$-t\,x(t) \longleftrightarrow \frac{dX(s)}{ds}$$

$$(-t)^2 x(t) \longleftrightarrow \frac{d^2X(s)}{ds^2}$$

$$L^{-1}\{X(s)\} = L^{-1}\Big\{\underbrace{s}_{X_1(s)} \cdot \underbrace{\frac{d^2}{ds^2}\Big(\frac{1}{s^2+9}\Big)}_{X_2(s)}\Big\} + L^{-1}\{1/s+3\}$$

$$\frac{d^2}{ds^2} X_2(s) \longleftrightarrow (-t)^2 x_2(t)$$

$$\longleftrightarrow t^2 \cdot \frac{1}{3}\sin 3t \, u(t)$$

$$X_2(s) \longleftrightarrow x_2(t)$$

$$S \cdot X_2(s) \longleftrightarrow \frac{dx_2(t)}{dt}$$

$$\Rightarrow \frac{dx_2(t)}{dt} = \frac{d}{dt}\Big(\frac{1}{3} t^2 \sin 3t \, u(t)\Big)$$

$$= \frac{1}{\cancel{3}} t^2 \cos 3t \, \cancel{3} u(t) + \frac{2t}{3}\sin 3t \, u(t).$$

$$= t^2 \cos 3t \, u(t) + \frac{2}{3} t \sin 3t \, u(t)$$

$$x(t) = t^2 \cos 3t \, u(t) + \frac{2}{3} t \sin 3t \, u(t) + e^{-3t} u(t)$$

Find $L\{e^{-a|t|}\}$.

$$-a < Re\{s\} < a$$

$x(t) = e^{-at}$ for $t > 0$.
$= e^{at}$ for $t < 0$.

$$\Rightarrow X(s) = \int_{-\infty}^{0} e^{at} e^{-st} dt + \int_{0}^{\infty} e^{-at} e^{-st} dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_{0}^{\infty}$$

$$= \frac{1}{a-s} + \frac{1}{a+s} \Rightarrow \frac{a+\cancel{s}+a-\cancel{s}}{a^2-s^2} = \frac{2a}{a^2-s^2} = \frac{-2a}{s^2-a^2}$$

$a-s > 0 \Rightarrow \{s\} < a$ $\quad$ $a+s > 0 \Rightarrow Re\{s\} > -a$

Scanned with OKEN Scanner

→ Find the initial and final values of x(t) if

$$X(s) = e^{-5s}\left(\frac{-2}{s(s+2)}\right)$$

Initial value $x(0^+) = \underset{s\to\infty}{Lt}\; sX(s)$

$$= \underset{s\to\infty}{Lt}\; \cancel{s}\, e^{-5s}\left(\frac{-2}{\cancel{s}(s+2)}\right)$$

$$= e^{-\infty}\left(\frac{-2/\infty}{1+2/\infty}\right)$$

$$= 0.$$

Final-value ⇒ $x(\infty) = \underset{s\to 0}{Lt}\; sX(s)$

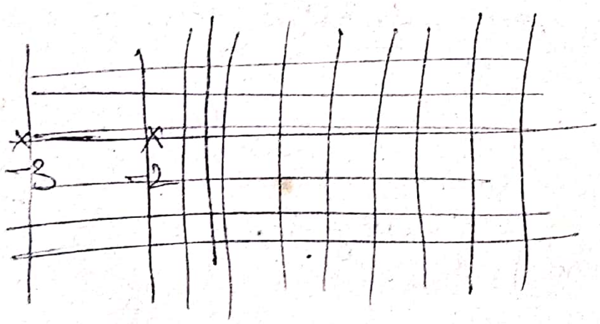
$$= \underset{s\to 0}{Lt}\; e^{-5s}\left(\frac{-2}{s+2}\right)$$

$$= \frac{-2}{2} = -1$$

→ $x(t) = e^{-3t}u(t) + e^{-2t}u(t)$ and find X(s) and ROC

$$X(s) = \frac{1}{s+3} + \frac{1}{s+2}$$

$Re\{s\} > -3$ $Re\{s\} > -2$



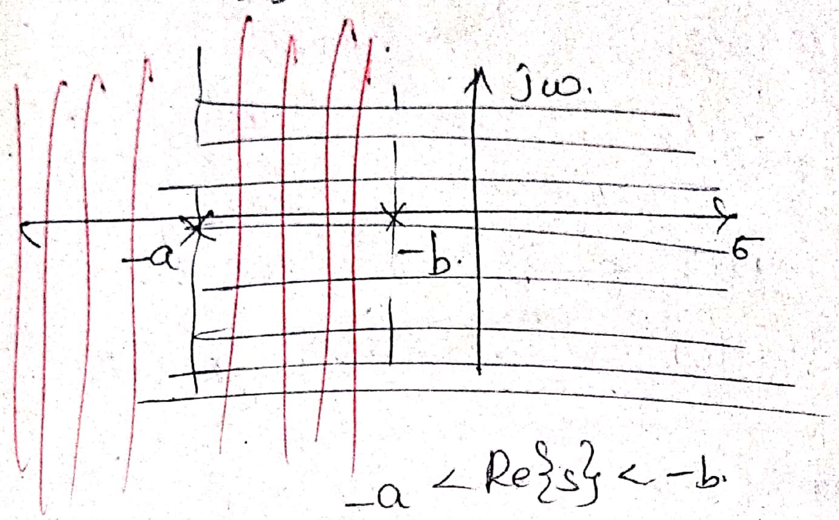
ROC : $Re\{s\} > -2$

→ Find the L.T/F of the signal

$$x(t) = e^{-at}u(t) + e^{-bt}u(-t)$$

$$X(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

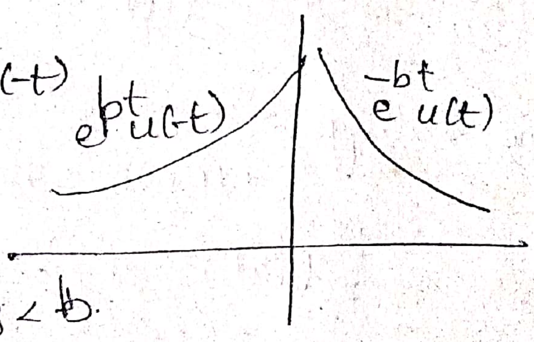
$Re\{s\} > -a \quad Re\{s\} < -b$



$-a < Re\{s\} < -b$

→ Find the LT of $x(t) = e^{-b|t|}$

$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$

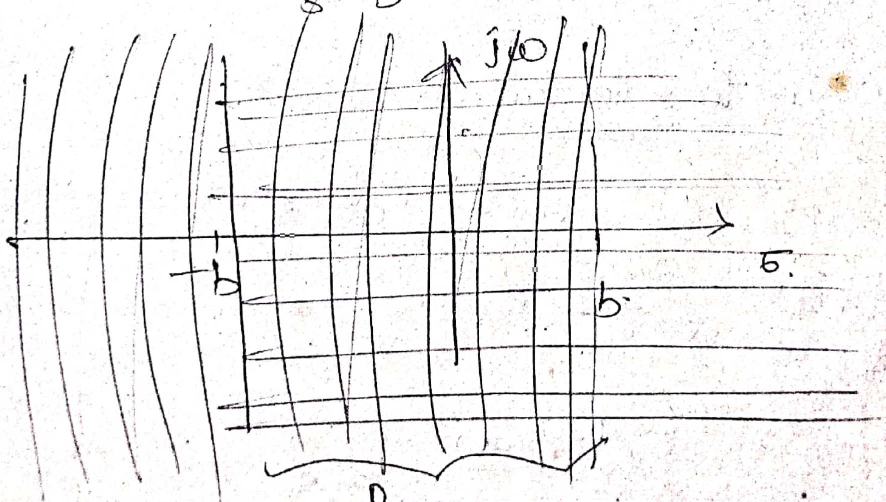


$$X(s) = \frac{1}{s+b} - \frac{1}{s-b}$$

$Re\{s\} > -b \quad Re\{s\} < b$

$$= \frac{s-b-s-b}{s^2-b^2}$$

$$= \frac{-2b}{s^2-b^2} \qquad b < Re\{s\} < -b$$



$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds \qquad X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Properties

Linearity:- $a\,x_1(t) + b\,x_2(t) \overset{L}{\longleftrightarrow} aX_1(s) + bX_2(s)$

$x_1(t)$: R_1, $x_2(t)$: R_2 ROC: $R_1 \cap R_2$

Time-shifting

$x(t) \longleftrightarrow X(s)$, ROC: R

$x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$. ROC: R

Shifting in s-domain

$x(t) \longleftrightarrow X(s)$, ROC: R

$e^{s_0t}x(t) \longleftrightarrow X(s-s_0)$ ROC: $R + Re\{s_0\}$

Time-Scaling:-

$x(t) \longleftrightarrow X(s)$, ROC: R.

$x(at) \longleftrightarrow \frac{1}{|a|}X(s/a)$, ROC: $\pm R/a$

Time-Reversal:-

$x(t) \longleftrightarrow X(s)$, ROC: R.

$x(-t) \longleftrightarrow X(-s)$, ROC: -R.

Differentiation in time domain

$x(t) \longleftrightarrow X(s)$, ROC: R.

$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s)$, ROC: R.

Differentiation in s-domain:-

$x(t) \longleftrightarrow X(s)$, ROC: R

$(-t)^n x(t) \longleftrightarrow \frac{d^n X(s)}{ds^n}$, ROC: R.

Integration in time-domain (Bi-lateral)

$$x(t) \longleftrightarrow X(s), \quad ROC: R$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s}, \quad ROC: R \cap [Re\{s\} > 0]$$

$$\int_{-\infty}^{t}\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} x(\tau)\,d\tau_1\,d\tau_2 \ldots \longleftrightarrow \frac{X(s)}{s^n}$$

Integration in - s-domain:- (Bi-lateral)

$$x(t) \longleftrightarrow X(s), \quad ROC: R.$$

$$\frac{x(t)}{t} \longleftrightarrow \int_{s}^{\infty} X(s)\,ds, \quad ROC: R$$

Convolution

$$\underset{R_1}{x(t)} * \underset{R_2}{h(t)} \longleftrightarrow X(s)H(s) \quad ROC: R_1 \cap R_2$$

Integration in - time domain (Uni-lateral)

$$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)\,d\tau}{s}.$$

Initial-value Theorem:-

$$x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)$$

Final-value Theorem:-

$$x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

$$\frac{d^n x(t)}{dt^n} = s^n X(s) - s^{n-1} x(t)\Big|_{t=0^-} - s^{n-2}\frac{dx(t)}{dt}\Big|_{t=0^-} - \cdots$$

$$\cdots \frac{d^{n-1}}{dt^{n-1}} x(t)\Big|_{t=0^-}$$

Related Formulas (Laplace transforms)

Signal	L.T/F	ROC
1. $e^{at}u(t) \rightarrow$	$\frac{1}{s-a}$	$Re\{s\} > a$
2. $e^{-at}u(t) \rightarrow$	$\frac{1}{s+a}$,	$Re\{s\} > -a$
3. $e^{at}u(-t) \rightarrow$	$\frac{-1}{s-a}$,	$Re\{s\} < a$
4. $e^{-at}u(-t) \rightarrow$	$\frac{-1}{s+a}$,	$Re\{s\} < -a$
$\delta(t)$	1	entire s-plane.
$u(t)$	$1/s$	$Re\{s\} > 0$
$t\,u(t)$	$1/s^2$	$Re\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$Re\{s\} > 0$
$\cos at\, u(t)$	$\frac{s}{s^2+a^2}$	$Re\{s\} > 0$
$\sin at\, u(t)$	$\frac{a}{s^2+a^2}$	″
$e^{-bt}\cos at\, u(t)$	$\frac{s+b}{(s+b)^2+a^2}$	″
$e^{-bt}\sin at\, u(t)$	$\frac{\cancel{s+b}\; a}{(s+b)^2+a^2}$	″
$\sinh at\, u(t)$	$\frac{a}{s^2-a^2}$	″
$\cosh at\, u(t)$	$\frac{s}{s^2-a^2}$	″
$t^n e^{at} u(t)$	$\frac{n!}{(s-a)^{n+1}}$	$Re\{s\} > 0$

$e^{-a|t|} \xrightarrow{F.T/F} \frac{2a}{\omega^2+a^2}$

$\xleftrightarrow{L.T/F} \frac{2a}{a^2-s^2}$ $\quad -a < Re\{s\} < a$

$t \sin at\, u(t) \longleftrightarrow \frac{2as}{s^2+a^2}$ $\quad Re\{s\} > 0$.

→ a stable system's ROC includes $j\omega$ axis.

→ a Causal system's ROC is right of the right most pole.

→ Relation LT/F to FourierT/F

$$X(j\omega) = X(s)\big|_{s=j\omega} \quad \sigma=0$$